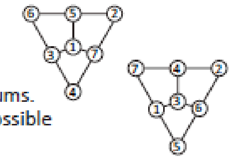
Warm-Up 17



241. There are four ways to get a sum of 15 using four different whole numbers from 1 to 7. They are [7 + 5 + 2 + 1], [7 + 4 + 3 + 1], [6 + 5 + 3 + 1] and [6 + 4 + 3 + 2]. The first three sums can be arranged together, as can the last three sums. If we use the first three, then A is replaced with 1. If we use the last three, then A is replaced with 3. The sum of all possible replacements for A is 1 + 3 = **4**.

242. Combined, the area of the shaded regions in the left upper and lower 4-by-4 squares is equal to the area of one 4-by-4 square, which is $4 \times 4 = 16$ units^2. The shaded region of the lower right 4-by-4 square is $(3/4) \times 16 = 12$ units^2. Finally, the circle in the upper right 4-by-4 square has a radius of $(1/2) \times 4 = 2$ units and an area of $\pi \times 2^2 = 4\pi$. Thus, the area of the shaded region surrounding the circle is $16 - 4\pi$ units. Therefore, the total area of the shaded regions is $16 + 12 + (16 - 4\pi) =$ **$44 - 4\pi$ units^2**.

243. The mean of an arithmetic sequence is the middle number when there are an odd number of terms, so we know the middle number is 18. Let's call the common difference between terms d. Then the five numbers are $18 - 2d$, $18 - d$, 18, $18 + d$ and $18 + 2d$. The sum of the squares of these five numbers is $(324 - 72d + 4d^2) + (324 - 36d + d^2) + 324 + (324 + 36d + d^2) + (324 + 72d + 4d^2) = 5 \times 324 + 10d^2$. We divide this by 5 to get a mean of $324 + 2d^2$, which is known to be 374. This means that $2d^2 = 374 - 324 \rightarrow 2d^2 = 50 \rightarrow d^2 = 25 \rightarrow d = 5$. The largest of the five original numbers is $18 + 2 \times 5 =$ **28**.

244. We need to determine how many sets of prime numbers, not necessarily distinct, have a sum of 13. We need to consider sets containing prime numbers between 2 and 13. The 9 numbers for which the sum of the prime factors is 13 are shown.

13
22 = 2 × 11
56 = 2 × 2 × 2 × 7
63 = 3 × 3 × 7
75 = 3 × 5 × 5
80 = 2 × 2 × 2 × 2 × 5
90 = 2 × 3 × 3 × 5
96 = 2 × 2 × 2 × 2 × 2 × 3
108 = 2 × 2 × 3 × 3 × 3

245. Let's work this problem backward. At the end, Xavier, Yvonne and Zeena all have 48 marbles, so let $x = 48$, $y = 48$ and $z = 48$. Before Zeena doubled Xavier's and Yvonne's marbles, it was $x = 24$, $y = 24$ and $z = 96$. Before Yvonne doubled Xavier's and Zeena's marbles, it was $x = 12$, $y = 84$ and $z = 48$. Before Xavier doubled Yvonne's and Zeena's marbles, the starting quantities were $x = 78$, $y = 42$ and $z = 24$. Xavier had 78 – 48 = **30** fewer marbles at the end than he started with.

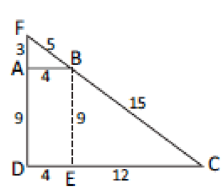
246. The $14 represents the extra 7 – 5 = 2 parts of the phone bill that Barbara had beyond Tina's phone bill. If $14 was 2 parts, then Barbara's bill must have been 7 × $7 = **$49**.

247. The least possible sum of four different positive integers is 1 + 2 + 3 + 4 = 10. To get to a sum of 13, we have to add 3 more to these numbers in such a way that they remain different. There are only 3 ways this can be done. They are [1 + 2 + 3 + 7], [1 + 2 + 4 + 6] and [1 + 3 + 4 + 5]. For each of these possible sums, there are 4! = 4 × 3 × 2 × 1 = 24 ways the numbers can be placed in the 4 boxes. Thus, there are a total of 3 × 24 = **72** ways in which four different positive integers with a sum of 13 can be arranged in the boxes.

248. Solving this problem requires some tricky algebra. Let the three numbers be a, b and c. From the information given, we have $a + b + c = 5$, $a^2 + b^2 + c^2 = 29$ and $abc = -10$. If we square each side of the first equation, we get $(a + b + c)^2 = 5^2 \rightarrow a^2 + b^2 + c^2 + 2ab + 2bc + 2ac = 25$. Now let's substitute 29 for $a^2 + b^2 + c^2$ in this equation to get $29 + 2ab + 2bc + 2ac = 25$. Subtracting 29 from each side and dividing the equation by 2 yields $ab + bc + ac = -2$. Let's now multiply each side of this equation by c to get $abc + bc^2 + ac^2 = -2c$. We can replace abc with -10 to get $-10 + bc^2 + ac^2 = -2c$. Now let's factor out c^2 on the left side of the equation to get $-10 + c^2(b + a) = -2c$. If $a + b + c = 5$ then $a + b = 5 - c$. Thus, we can replace $b + a$ with $5 - c$ to get $-10 + c^2(5 - c) = -2c$. Distributing c^2 and combining like terms yields $-10 = c^3 - 5c^2 - 2c$. If we set this cubic equal to 0, the result is $c^3 - 5c^2 - 2c + 10 = 0$. Factoring this equation yields $c^2(c - 5) - 2(c - 5) = 0 \rightarrow (c^2 - 2)(c - 5) = 0$. Now we see that the three solutions are $-\sqrt{2}$, $\sqrt{2}$ and 5, the least of which is **$-\sqrt{2}$**.

249. Let's say the total value of the quarters and dimes is T. Then $T = 25q + 10d$, where q is the number of quarters and d is the number of dimes. When 10% more quarters are added, the value of T increases by 7.5%, so $1.075T = 1.1 \times 25q + 10d$. Let's rewrite this with fractions. Ten percent is 1/10 and 7.5% is 3/40, so we have the following equation: $(43/40)T = (11/10) \times 25q + 10d$. Substituting the original value of T, we have $(43/40)(25q + 10d) = (11/10)(25q) + 10d$. Multiplying out both sides and simplifying, we get $(215/8)q + (43/4)d = (55/2)q + 10d \rightarrow (43/4)d - (40/4)d = (220/8)q - (215/8)q \rightarrow (3/4)d = (5/8)q \rightarrow q/d = (3/4)/(5/8) \rightarrow q/d = (3/4) \times (8/5) =$ **6/5**.

250. After transforming the trapezoid, the resulting shape is a frustum, which is simply a cone with the tip removed. We can determine the volume of the frustum by determining the volume of the cone less the volume of the portion that is removed to form the frustum. If we drop a perpendicular from point B down to $\overline{CD}$, as shown, and label the point of intersection E, then we have right ΔBCE with hypotenuse of length 15 units and one leg of length 16 – 4 = 12 units. This is a multiple of the 3-4-5 Pythagorean triple 9-12-15, so the length of segment BE must be 9 units. If we extend $\overline{AD}$ and $\overline{BC}$ until they intersect at point F, we create ΔFCD and ΔFBA with sides of length 12 units, 16 units and 20 units, and 3 units, 4 units and 5 units, respectively. (This is because AB = 4 and $\Delta BCE \sim \Delta FBA$.) Now let's revolve ΔFCD around $\overline{FD}$. Recall that the volume of a cone is 1/3 the product of the cone's height and the area of its base. Therefore, the volume of this cone is $1/3 \times 12 \times \pi \times 16^2 = 1024\pi$ units^3. Next we revolve ΔFBA around $\overline{FA}$. The volume of this cone is $1/3 \times 3 \times \pi \times 4^2 = 16\pi$ units^3. Thus, the volume of the frustum is $1024\pi - 16\pi =$ **1008π units^3**.



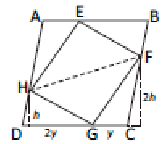
Warm-Up 18

251. It is safe to assume that the greatest possible product will be produced by a two-digit number times a three-digit number, so let's call those numbers AB and XYZ. The value of AB is 10A + B and the value of XYZ is 100X + 10Y + Z. The product (10A + B)(100X + 10Y + Z) = 1000AX + 100AY + 100BX + 10AZ + 10BY + BZ. First let's compare the way the digits A and X are used. They each get multiplied by 1000 and by each other, which is symmetric. They also each get multiplied by 100 and by another digit, but A is also multiplied by 10Z; therefore we want A to be the largest number of the set, and X should be the next largest. Thus, we determine that A = 8 and X = 7. We can now write (1000 × 8 × 7) + (100 × 8 × Y) + (100 × 7 × B) + (10 × 8 × Z) + 10BY + BZ = 56,000 + 800Y + 700B + 80Z + 10BY + BZ. Now, we can see that to get the largest product, Y should be the next greatest value, followed by B and then Z. So we have Y = 5, B = 3 and Z = 1. The two numbers are 83 and 751, and their product is **62,333**.

252. Since the total area is 400 ft^2 and the ratio of the areas of the two smaller rectangles is 3 to 1, they must have areas of 300 ft^2 and 100 ft^2. The least possible perimeter for the smaller rectangle would be 4 × 10 = **40** ft, from a 10 × 10 square.

253. A set of seven positive integers with a mean of 13 must have a sum of 7 × 13 = 91. Since the integers must be positive and different, the smallest possible value of the median is 4. The numbers could be 1, 2, 3, 4 and any combination of three different positive integers adding up to 91 – (1 + 2 + 3 + 4) = 81. To find the largest possible median, we should first allow for 1 + 2 + 3 = 6 at the low end. That leaves 91 – 6 = 85 to be distributed to the other four integers. We will need another 1 + 2 + 3 = 6 to make the 5th, 6th and 7th integers greater than the 4th integer, so that leaves 85 – 6 = 79 to divide by 4. Algebraically, we can solve $n + (n + 1) + (n + 2) + (n + 3) < 85$ for the greatest integer value of n that satisfies the inequality. This simplifies to $4n < 79$, and 19 is the greatest integer value of n. The desired difference is 19 – 4 = **15**.

254. The value of 123 (base b) is $1 \times b^2 + 2 \times b + 3 \times 1$ and this must equal 363. Algebraically, this can be written as $b^2 + 2b + 3 = 363$. We set the equation equal to zero to get $b^2 + 2b - 360 = 0$. To factor this quadratic, we look for two factors of 360 that differ by 2. They are 18 and 20, so the equation can be written as $(b + 20)(b - 18) = 0$. The solutions to this are $b = -20$ and $b = 18$. Only the positive solution makes sense here, so the base must be **18**.

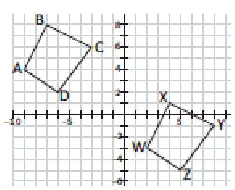


255. Let GC = y, so DG = $2y$ and DC = $3y$. Now let the length of the altitude of ΔDHG be h. Then the altitude of ΔGCF has length $2h$ and the altitude of rhombus ABCD has length $3h$. The area of rhombus ABCD is then $3y \times 3h = 9yh$ and is known to be 180 $units^2$. This means that $yh = 180 \div 9 = 20$. The area of ΔDHG is $1/2 \times 2y \times h = yh = 20$ $units^2$, and the area of ΔGCF is $(1/2) \times y \times 2h = yh = 20$ $units^2$. The areas of ΔAEH and ΔEBF are also each 20 $units^2$. This means that rhombus EFGH must have an area of 180 – (4 × 20) = **100** $units^2$.

256. Let's substitute the solution $x = u$ into the first equation and $x = 1/u$ into the second equation. The resulting equations are $ru^2 + su + t = 0$ and $(2 + a)/u^2 + 5/u + (2 - a) = 0$. We rewrite the second equation with a common denominator and get $(2 + a)/u^2 + 5u/u^2 + [(2 - a)u^2]/u^2 = 0$. Multiplying both sides of the equation by u^2, we get $(2 + a) + 5u + (2 - a)u^2 = 0$. Since our first equation and this last equation are both equal to zero, we can set them equal to each other to get $ru^2 + su + t = (2 + a) + 5u + (2 - a)u^2$. Both of these expressions are in terms of u, so the coefficients of the same degree must be equal. This means that $r = 2 - a$, $s = 5$ and $t = 2 + a$. Thus, the sum $r + s + t$ is $2 - a + 5 + 2 + a =$ **9**.

257. There are ${}_{22}C_4 = \frac{22 \times 21 \times 20 \times 19}{4 \times 3 \times 2 \times 1}$ ways to choose a committee of 4 from a group of 22. Similarly, there are ${}_{22}C_5 = \frac{22 \times 21 \times 20 \times 19 \times 18}{5 \times 4 \times 3 \times 2 \times 1}$ ways to choose a committee of 5 from a group of 22. The ratio of the number of ways to choose a 4-person committee to the number of ways to choose a 5-person committee is $\frac{(22 \times 21 \times 20 \times 19)/(4 \times 3 \times 2 \times 1)}{(22 \times 21 \times 20 \times 19 \times 18)/(5 \times 4 \times 3 \times 2 \times 1)} = \frac{\cancel{22} \times \cancel{21} \times \cancel{20} \times \cancel{19}}{\cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} \times \frac{5 \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{22} \times \cancel{21} \times \cancel{20} \times \cancel{19} \times 18} = \mathbf{\frac{5}{18}}$.

258. Quadrilateral WXYZ is a translation of ABCD 11 units to the right and 7 units down, as shown. The coordinates of point Z are (5, –5), so the sum of the coordinates is 5 + (–5) = **0**.



259. Substituting a and $c = 49 - a$ into the Pythagorean Theorem and simplifying yields $(49 - a)^2 - a^2 = b^2 \rightarrow 2401 - 98a + a^2 - a^2 = b^2 \rightarrow 2401 - 98a = b^2 \rightarrow 49(49 - 2a) = b^2$. This means that $49 - 2a$ is an odd, square number, in which case the candidates are 1, 9, 25 and 49. If $49 - 2a = 1$, solving for a yields $-2a = -48 \rightarrow a = 24$. So we have $a = 24$, $b = 7$ and $c = 25$. For this particular triangle $b < a < c$, and this will not satisfy our conditions. If $49 - 2a = 9$, solving for a yields $-2a = -40 \rightarrow a = 20$. So we have $a = 20$, $b = 21$ and $c = 29$. The area of this triangle is (1/2) × 20 × 21 = 210 $units^2$. If $49 - 2a = 25$, solving for a yields $-2a = -24 \rightarrow a = 12$. So we have $a = 12$, $b = 35$ and $c = 37$. The area of this triangle is (1/2) × 12 × 35 = 210 $units^2$. Finally, $49 - 2a = 49$ doesn't work since it results in $a = 0$. The area of each of the two triangles that satisfy our conditions is **210** $units^2$.

260. This problem may prove easier to solve if we can situate the figure in the coordinate plane. Let's suppose the left, rear corner of the base, which is point E, is situated at the origin with coordinates (0, 0, 0). Since all the edges have length 2 units, the other vertices would be located as follows: F(2, 0, 0), G(2, 0, 2), H(0, 0, 2), A(0, 2, 0), B(2, 2, 0), C(2, 2, 2), D(0, 2, 2), M(0, 0, 1) and N(1, 0, 0). If we let point K be the midpoint of $\overline{MN}$, the original tetrahedron is divided into two congruent tetrahedrons with common base ΔACK. Since $\overline{MN}$ is the hypotenuse of an isosceles right triangle with legs of length 1 unit, its length is √2 units. Thus, each of these two smaller tetrahedrons has height h = MK = NK = √2/2 units. Now to determine the area of ΔACK, we first consider rectangle ACGE. Since $\overline{AC}$ is the diagonal of one of the faces of the cube, it is also the hypotenuse of a 45-45-90 right triangle with legs of length 2 units. That means AC = 2√2 units and rectangle ACGE has an area of 4√2 $units^2$. Next we determine the area of ΔAEK to be (1/2) × (√2/2) × 2 = √2/2 $units^2$, and the area of ΔCGK is (1/2) × (3√2/2) × 2 = 3√2/2 $units^2$. If we subtract these two areas from the area of rectangle ACGE, the result is the area of ΔACK. The area of ΔACK is 4√2 – (√2/2) – (3√2/2) = (8√2/2) – (√2/2) – (3√2/2) = (4√2/2) = 2√2 $units^2$. The volume of each of the smaller tetrahedrons is (1/3) × (2√2) × (√2/2) = 2/3 $units^3$. Therefore the volume of the large tetrahedron is 2/3 + 2/3 = **4/3** $units^3$.